Sample: Real Analysis - Real Analysis Task

Problem 1. Let \( \{x_n\} \) and \( \{y_n\} \) be sequences of real numbers with the property
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = L \in [-\infty, +\infty].
\]
Prove that the sequences \( a_n = \frac{x_n + y_n}{2} \) and \( b_n = \sqrt{x_n y_n} \) are convergent to \( L \).
(For \( \{b_n\} \), it is assumed that \( x_n y_n \geq 0 \), of course.)

Solution. Suppose \( \varepsilon > 0 \).

Since we are given \( \lim_{n \to \infty} x_n = L \), then by definition of the limit of a sequence there exists some \( N_1 \in \mathbb{N} \) such that if \( n > N_1 \) then \( |x_n - L| < \frac{\varepsilon}{2} \).

Similarly, since \( \lim_{n \to \infty} y_n = L \), there exists some \( N_2 \in \mathbb{N} \) such that if \( n > N_2 \) then \( |y_n - L| < \frac{\varepsilon}{2} \).

Let us now choose \( N = \max\{N_1, N_2\} \). For \( n > N \), both of the above inequalities hold. We can therefore add them together:
\[
|x_n - L| + |y_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
for every \( n > N \).

Next, recalling the triangle inequality, we have
\[
|(x_n + y_n) - (L + L)| \leq |x_n - L| + |y_n - L|,
\]
and thus
\[
|(x_n + y_n) - 2L| < \varepsilon.
\]

We see that \( \lim_{n \to \infty} (x_n + y_n) = 2L \).

Finally, recall the following property of limits of real sequences:
\[
\lim_{n \to \infty} c x_n = c \lim_{n \to \infty} x_n \quad \text{for every } c \in \mathbb{R}.
\]

By applying this property, we have
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2} (x_n + y_n) = \frac{1}{2} \lim_{n \to \infty} (x_n + y_n) = \frac{1}{2} \cdot 2L = L,
\]
and we have completed the first part of the proof.

Let us now look at sequence \( b_n \).

This part will be slightly more complicated, since we will need to use an additional result: every convergent sequence is bounded. Let us prove this statement.

We will use \( \{x_n\} \) as an example. Recall that we have \( \lim_{n \to \infty} x_n = L \in \mathbb{R} \), which means that \( \{x_n\} \) is a convergent sequence. According to the definition given above, there exists some \( N_1 \in \mathbb{N} \) such that if \( n > N_1 \) then \( |x_n - L| < \epsilon \), or \( L - \epsilon < x_n < L + \epsilon \).

The set \( \{x_n: 1 \leq n \leq N_1\} \) is finite and therefore bounded: there exist \( m, M \in \mathbb{R} \) such that for all \( n \leq N_1 \), we have \( m < x_n < M \).

Now take \( m' = \min\{L - \epsilon, \ m\} \) and \( M' = \max\{L + \epsilon, M\} \). We now have that for all \( n \in \mathbb{N}, \ m' < x_n < M' \), so the sequence \( \{x_n\} \) is bounded.

We can now proceed to the final part of our proof.

Applying the result above to both \( \{x_n\} \) and \( \{y_n\} \), we can state that there exist \( m', m'', M', M'' \in \mathbb{R} \) such that for all \( n \in \mathbb{N}, \ m' < x_n < M' \) and \( m'' < y_n < M'' \). We now choose \( M = \max\{1, |L|, |m'|, |m''|, |M'|, |M''|\} \).

We can repeat our definition of convergence for \( x_n \) and \( y_n \):

- there exists some \( N_1 \in \mathbb{N} \) such that if \( n > N_1 \) then \( |x_n - L| < \frac{\epsilon}{2M} \);
- there exists some \( N_2 \in \mathbb{N} \) such that if \( n > N_2 \) then \( |y_n - L| < \frac{\epsilon}{2M} \).

Just like for \( a_n \), choose \( N = \max\{N_1, N_2\} \), and both inequalities hold for \( n > N \).

Now let us evaluate \( |x_n y_n - L^2| \).
\[ |x_ny_n - L^2| = |x_ny_n - x_nL + x_nL - L^2| = |x_n(y_n - L) + L(x_n - L)| \]

Apply the triangle inequality:
\[ |x_n(y_n - L) + L(x_n - L)| \leq \cdots \]

situated on the real line, respectively on the \( y \)-axis. These points are \((2, 0), (0, 2), (-2, 0), (0, -2)\).

Now consider
\[ \lim_{n \to \infty} x_n = 2 \]

To find the radius of convergence, we will first need to find the limit superior of the sequence \( \{c_n\} \), where (in our case) \( c_n = \frac{1 + (-1)^n}{\sqrt{n}} \). To do this, note that \( c_{2k} = \frac{1 + 1}{\sqrt{2k}} = \frac{2}{\sqrt{2k}} > 0 \), whereas
\[ c_{2k+1} = \frac{1 - 1}{\sqrt{2k+1}} = 0. \]

Thus,
\[ \lim_{n \to \infty} \sup_{n \geq 0} \left( \frac{1 + (-1)^n}{\sqrt{n}} \right)^\frac{1}{n} = \lim_{n \to \infty} \left( \frac{1}{\sqrt{2n}} \right)^\frac{1}{n} = \lim_{n \to \infty} \frac{1}{2\sqrt{\sqrt{n}}} = 1. \]

Now note that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = \lim_{n \to \infty} e^{\frac{1}{2} \ln n} = e^{\frac{1}{2}} = 1 \) and \( \lim_{n \to \infty} 2^n = 2^0 = 1 \). Therefore,
\[ \lim_{n \to \infty} \sup_{n \geq 0} \left( \frac{1 + (-1)^n}{\sqrt{n}} \right)^\frac{1}{n} = 1 \]

Applying the formula for radius of convergence, we have
\[ R = \frac{1}{\lim_{n \to \infty} \sup_{n \geq 0} \left( \frac{1 + (-1)^n}{\sqrt{n}} \right)^\frac{1}{n}} = 1. \]

The disk of convergence is \( \Delta_R = \{ z : |z| < 2 \} \).

The last part of the problem is to study the convergence at the points \( z \) on the boundary of that disk situated on the real line, respectively on the \( y \)-axis. These points are \((2, 0), (0, 2), (-2, 0), (0, -2)\).

- \( z_1 = 2 \)
$$\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n} 2^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Note that the series $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ diverges, since we know that power series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$ and in our case $p = \frac{1}{2} < 1$.

However, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges, which can be shown by the alternating series test, since the sequence $\left(\frac{1}{\sqrt{n}}\right)$ decreases monotonically and goes to zero in the limit as $n \to \infty$.

Now recall that the sum of a convergent and divergent series diverges. Thus, $\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n} 2^n}$ diverges.

- $z_2 = 2i$
  
  Recall that $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$, $i^{4k} = 1$ for every $k \in \mathbb{N}$. Therefore,
  $$\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n} 2^n} (2i)^n = \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n+1}}{\sqrt{4n+1}} i - \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n+2}}{\sqrt{4n+2}} - \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n+3}}{\sqrt{4n+3}} i$$
  $$+ \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n}}{\sqrt{4n}} = i \sum_{n=0}^{\infty} \left( \frac{1 - 1}{\sqrt{4n+1}} - \frac{1 - 1}{\sqrt{4n+3}} + \frac{1 + 1}{\sqrt{4n}} - \frac{1 + 1}{\sqrt{4n+2}} \right)$$
  $$+ \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n+2}}{\sqrt{4n+2}}$$

  Now note that $(-1)^{4n} = (-1)^{4n+2} = 1$ and $(-1)^{4n+1} = (-1)^{4n+3} = -1$. Thus, we can further simplify this expression as follows:
  $$\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n} 2^n} (2i)^n = i \sum_{n=0}^{\infty} \left( \frac{1 - 1}{\sqrt{4n+1}} - \frac{1 - 1}{\sqrt{4n+3}} + \frac{1 + 1}{\sqrt{4n}} - \frac{1 + 1}{\sqrt{4n+2}} \right)$$
  $$= \sum_{n=0}^{\infty} \frac{2}{\sqrt{4n} - \sqrt{4n+2}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n} - \sqrt{n+2}}$$

  Let us now investigate convergence of this series. We will transform the summand:
  $$\frac{1}{\sqrt{n} - \sqrt{n+2}} = \frac{2}{\sqrt{4n+2} - \sqrt{4n+2} + 2\sqrt{n}} = \frac{1}{\sqrt{n+2} - \sqrt{n+2} + 2\sqrt{n}} = \frac{1}{2\sqrt{n+2}}$$

  We will use the comparison convergence test. Since $\sqrt{n+2} > \sqrt{n}$, we have
  $$\frac{2}{\sqrt{n+2} - \sqrt{n+2} + 2\sqrt{n}} \leq \frac{2}{\sqrt{n} \sqrt{n+2} + 2\sqrt{n}} = \frac{2}{2n \left( 2\sqrt{n} + 2\sqrt{n} \right)} = \frac{1}{3n^{\frac{3}{2}}}$$

  Recall again that the power series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$; here $p = \frac{3}{2} > 1$, so our series $\sum_{n=0}^{\infty} \frac{1}{3n^{\frac{3}{2}}}$ converges.

  Finally, by applying the comparison convergence test, we see that the initial series $\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n} 2^n} (2i)^n$ also converges.

- $z_3 = -2$
\[
\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n} 2^n} \left(-2\right)^n = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n}} - \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}
\]

This expression is equal to the one we obtained for \( \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n} 2^n} \), and we have already shown that this series diverges above.

- \( z_4 = -2i \)

\[
\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \left(-1\right)^n \frac{1}{\sqrt{n}}
\]

We will use the same approach as for \( z_2 = 2i \).

\[
\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n}} \left(-i\right)^n = -\sum_{n=0}^{\infty} \frac{1 + (-1)^{4n+1}}{\sqrt{4n+1}} i - \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n+2}}{\sqrt{4n+2}} i + \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n+3}}{\sqrt{4n+3}} i
\]

\[
= -\sum_{n=0}^{\infty} \frac{1 + (-1)^{4n}}{\sqrt{4n}} i + \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n}}{\sqrt{4n}} i
\]

\[
= \sum_{n=0}^{\infty} \frac{1 + (-1)^{4n}}{\sqrt{4n}} i
\]

**Problem 3.** Let \( f: \mathbb{R} \to \mathbb{R} \) be a continuous function with the property \( \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty \). Prove that such a function attains its minimum.

**Solution.** Let us first consider what we mean by \( \lim_{x \to +\infty} f(x) = +\infty \): for every \( M > 0 \), there exists some \( n_1 > 0 \) such that for all \( x > n_1 \), we have \( f(x) > M \).

Similarly, \( \lim_{x \to -\infty} f(x) = +\infty \) is equivalent to the statement that for every \( M > 0 \), there exists some \( n_2 > 0 \) such that for all \( x < -n_2 \), \( f(x) > M \).

Therefore, for every \( M > 0 \) we can choose \( n = \max\{n_1, n_2\} \) so that if \( |x| > n \), then \( f(x) > M \). We see that \( f \) does not attain its minimum outside \([-n, n]\).

But \([-n, n]\) is a compact set. Since the function \( f \) is continuous, it attains a minimum on \([-n, n]\) (by the Extreme Value Theorem). Let us denote the point where the minimum is attained as \( x_0 \): \( f(x_0) = \min_{x \in [-n, n]} f(x) \).

Due to the way we chose \( n \), \( f(x) < M \) for all \( x \in [-n, n] \); thus, \( f(x_0) < M \), and we see that \( f(x_0) = \min_{x \in \mathbb{R}} f(x) \). The proof is complete.

**Problem 4.** Given that \( f: \mathbb{R} \to \mathbb{R} \) is differentiable at 0 and \( f'(0) = 1 \), find

\[
\lim_{x \to 0} \frac{f(x) - f(-x)}{x}.
\]

Give reasons for your answer.

**Solution.** To find the value of our expression, we will somewhat transform it by adding and subtracting \( f(0) \):

\[
\lim_{x \to 0} \frac{f(x) - f(-x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0) + f(0) - f(-x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x} + \lim_{x \to 0} \frac{f(0) - f(-x)}{x}
\]

In the second expression, we can introduce a new variable \( w = -x \):

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x} + \lim_{x \to 0} \frac{f(0) - f(-x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x} + \lim_{w \to 0} \frac{f(w) - f(0)}{w}.
\]
Now recall the definition of the derivative of function $f$:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$ 

Thus,

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}.$$ 

This is exactly the expression we obtained above. So we can write

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} + \lim_{w \to 0} \frac{f(w) - f(0)}{w} = f'(0) + f'(0) = 2 \cdot f'(0).$$ 

Finally, since we are given $f'(0) = 1$, we can say that

$$\lim_{x \to 0} \frac{f(x) - f(-x)}{x} = 2 \cdot 1 = 2.$$ 

Answer. $\lim_{x \to 0} \frac{f(x) - f(-x)}{x} = 2.$