1. The normalization condition is \( \int |\psi(x)|^2 \, dx = 1 \). The wave-function for \(-\pi < x < \pi\) might be rewritten as \( \psi(x) = A (e^{ix} + e^{-ix}) = 2A \cos x \). Thus, according to normalization condition, \( 4A^2 \int_{-\pi}^{\pi} \cos^2 x \, dx = 4A^2 \int_{-\pi}^{\pi} \frac{1+\cos 2x}{2} \, dx = 4A^2 \left( \frac{x}{2} + \frac{\sin 2x}{4} \right) \bigg|_{-\pi}^{\pi} = 4A^2 \cdot \pi = 1 \). Hence, \( A = \frac{1}{2\sqrt{\pi}} \) and \( \psi(x) = \cos \frac{x}{\sqrt{\pi}} \) for \(-\pi < x < \pi\).

\[
P(0 < x < \frac{\pi}{8}) = \int_{0}^{\frac{\pi}{8}} |\psi(x)|^2 \, dx = \frac{1}{\pi} \int_{0}^{\frac{\pi}{8}} \cos^2 x \, dx = \frac{1}{\pi} \left( \frac{x}{2} + \frac{\sin 2x}{4} \right) \bigg|_{0}^{\frac{\pi}{8}} = \frac{1}{16} \left( \frac{\pi}{16} + \frac{1}{4\sqrt{2}} \right).
\]

2. Plugging in \( \Psi(x,t) = \psi(x)f(t) \) into time-dependent Schroedinger equation yields
\[
i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V(x) \psi.
\]
Dividing the last equation by \( \psi(x)f(t) \), obtain
\[
i\hbar \frac{\partial}{\partial t} \frac{\psi}{f} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}.\] Since the right side does not depend on \( t \) explicitly, and left side is function of \( t \), then
\[
i\hbar \frac{\partial}{\partial t} \frac{\psi}{f} = E = \text{const},
\]
which yields time-independent Schroedinger equation \( -\frac{\hbar^2}{2m} \psi'' + V(x) \psi = E \psi \).

3.

a) Since for certain \( n \), \( l=0..n-1 \), thus for \( n = 6 \), \( l = 0..5 \).
b) Since for certain \( l \), \( m_l = -l..l \), for \( l = 6 \), \( m_l = -6..6 \).
c) Knowing that for certain \( n \), \( l = 0..n-1 \), the smallest possible value of \( n \), for which \( l = 4 \) is \( n = 5 \).
d) Knowing that for certain \( l \), \( m_l = -l..l \), the smallest possible value of \( l \), for which \( m_l = 4 \) is \( l = 4 \).

4.

a) \[
<r> = \int_{0}^{\infty} r^2 |R(r)|^2 \, dr = \frac{4}{\alpha_0^3} \int_{0}^{\infty} r^3 e^{-\alpha_0 r} \, dr = \frac{2}{\alpha_0^3} \int_{0}^{\frac{\alpha_0 r}{3}} t^3 e^{-t} \, dt = \frac{a_0^2}{4} \Gamma(4) = \frac{a_0^2}{4} \cdot \frac{3}{2} = \frac{3}{2} a_0.
\]
b) The Coulomb potential is \( U = -\frac{\alpha}{r} \). Thus,
\[
<U> = -4 \frac{\alpha}{\alpha_0^3} \int_{0}^{\infty} r^2 e^{-\alpha_0 r} \, dr = \frac{\alpha}{\alpha_0^3} \int_{0}^{\infty} t^2 e^{-t} \, dt = \frac{\alpha}{4} \frac{\alpha_0^3}{2} \Gamma(2) = \frac{-\alpha}{a_0}.
\]