Sample: Differential Geometry - Mathematics Assignment

**Question 1.** Steiner’s Roman surface is defined as the image of the map $F: \mathbb{RP}^2 \to \mathbb{R}^3$ induced by the map $\hat{F}: S^2 \to \mathbb{R}^3$.

Show that $F$ fails to be an immersion at six points on $\mathbb{RP}^2$.

**Proof.** Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Then by definition the projective plane $\mathbb{RP}^2$ is the space of pairs of antipodal points of $S^2$, that is the factor-space of $S^2$ by the following equivalence relation:

$(x_1, x_2, x_3): (−x_1, −x_2, −x_3)$.

Let us prove that $\hat{F}$ induces a certain map $\mathbb{RP}^2 \to \mathbb{R}^3$.

Let $\alpha: S^2 \to \mathbb{RP}^2$ be the factor map.

Since $(−x_1)(−x_j) = x_i x_j$ it follows that

$\hat{F}(−x_1, −x_2, −x_3) = (−x_2(−x_3), −x_1(−x_3), −x_1(−x_2))$

$= (x_2 x_3, x_1 x_3, x_1 x_2)$

$= \hat{F}(x_1, x_2, x_3)$.

Thus $\hat{F}$ constant of equivalence class, and so it induces a map $F: \mathbb{RP}^2 \to \mathbb{R}^3$ such that $\hat{F} = F \circ \alpha$.

Now let us check that $F$ is an immersion. First we recall the definition of an immersion.

Let $M, N$ are smooth two manifolds, $f: M \to N$ be a $C^1$ map, and $x \in M$. Then $f$ is an immersion at $x$ if the tangent map $T_x f: T_x M \to T_{f(x)} N$ is injective. Suppose dim$M = m$ and dim$N = n$, and we choose local coordinates $(x_1, ..., x_m)$ on $M$ at $x$ and $(y_1, ..., y_n)$ on $N$ at $f(x)$. Then $f$ is an immersion at $x$ if the Jacobi matrix of $f$ at $x$ (consisting of partial derivatives of coordinate functions of $f$) has rank $m$.

Evidently, a composition of immersions is an immersion as well.

Notice that the factor map $\alpha: S^2 \to \mathbb{RP}^2$ is local diffeomorphism, so the tangent map $\alpha$ at each point $q \in S^2$ is an isomorphism, and so $\alpha$ is an immersion. Thus in order to find points on $\mathbb{RP}^2$ at which $F$ is not an immersion we should find points on $S^2$ at which $\hat{F}$ is not an immersion and take their images in $\mathbb{RP}^2$.

Moreover, we can extend $\hat{F}$ to the map $\mathbb{R}^3 \to \mathbb{R}^3$ by the same formula. Then the Jacobi matrix of $\hat{F}$ is equal to

$$J(\hat{F}) = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$$

and its determinant is

$$|J(\hat{F})| = \begin{vmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{vmatrix} = -x_1 x_2 x_3 - x_1 x_2 x_3 = -2 x_1 x_2 x_3.$$

Let $q = (x_1, x_2, x_3)$. Then $|J(\hat{F})(q)| \neq 0$ if and only if all coordinates $(x_1, x_2, x_3)$ are non-zero, i.e. the point $q$ does not belongs to the coordinate planes $xy$, $yz$, and $xz$. At each of these points the tangent map

$$T_q \hat{F}: T_q \mathbb{R}^3 \to T_{\hat{F}(q)} \mathbb{R}^3$$

is an isomorphism. In particular, if in addition $q \in S^2$, the restriction of $T_q \hat{F}$ to the tangent plane $T_q S^2$ is injective, whence $\hat{F}$ is an immersion at $q$. Therefore at the corresponding point $\alpha(q) \in \mathbb{RP}^2$ the map $F$ is an immersion as well.
Suppose one of coordinates of $q$ is zero. Not loosing generality assume that $x_1 = 0$. As $x_1^2 + x_2^2 + x_3^2 = 1$, it follows that $x_2^2 + x_3^2 = 1$, whence either $x_2$ or $x_3$ is non-zero.

Then the Jacobi matrix at $q$ is

$$J(\hat{F})(q) = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}$$

and its rank (as of a map $\mathbb{R}^3 \to \mathbb{R}^3$) is 2, as at least one of the following $2 \times 2$-minores is non-zero:

$$\begin{vmatrix} 0 & x_3 \\ x_3 & 0 \end{vmatrix} = -x_3^2, \quad \begin{vmatrix} 0 & x_2 \\ x_2 & 0 \end{vmatrix} = -x_2^2.$$

Now the us find intersection of the null space of matrix $J(\hat{F})(q)$ with the tangent space $T_q S^2$. Then the restriction of $\hat{F}$ to $S^2$ is an immersion if and only if that intersection is non-zero.

Suppose the tangent vector $\xi = (a, b, c) \in T_q \mathbb{R}^3$ belongs to the null space of $J(\hat{F})(q)$. Thus

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = J(\hat{F})(p) \cdot \xi = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} bx_3 + cx_2 \\ ax_2 \\ ax_3 \end{pmatrix}.$$

As either $x_2$ or $x_3$ is non-zero, it follows that $a = 0$ and $bx_3 + cx_2 = 0$. Whence the null space of $J(\hat{F})(q)$ is spanned by the following vector

$$\eta = \begin{pmatrix} 0 \\ -x_2 \\ x_3 \end{pmatrix}.$$

The restriction of $\hat{F}$ to $S^2$ at $q$ is not an immersion if and only if $\eta$ belongs to the tangent space $T_q S^2$ of $S^2$ at $q$. The latter condition means that $\eta$ is orthogonal to the vector $\hat{q} \in \mathbb{R}^3$, so their scalar product is zero:

$$\langle \eta, \hat{q} \rangle = 0 = (0, -x_2, x_3) \cdot (0, x_2, x_3) = 0 \cdot 0 - x_2 x_2 + x_3 x_3 = -x_2^2 + x_3^2.$$

It then follows that

$$x_2^2 = x_3^2, \quad x_2 = \pm x_3.$$

As $x_2^2 + x_3^2 = 1$, we obtain that

$$x_2^2 = x_3^2 = \frac{1}{2}, \quad x_2 = \pm \frac{1}{\sqrt{2}}, \quad x_3 = \frac{1}{\sqrt{2}}.$$

Thus there are the 4 points on $S^2$ with $x_0$ at which $\hat{F}$ is not an immersion:

$$X_1 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad X_2 = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

$$X_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad X_4 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Since $X_1 = -X_2$ they define the same point $a(X_1) = a(X_2)$ on $\mathbb{R}P^2$, and at this point the map $F$ is not an immersion. The same statement hold for the pair $X_3$ and $X_4$.

Thus we have found two points on $PR^2$ with $x_1 = 0$ at which $F$ is not an immersion. Due to the symmetry, in each of the cases $x_2 = 0$ and $x_3 = 0$ we also have 2 non-immersion points, and so the map $F$ has the following six points at which it is not an immersion:

$$\pm \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \pm \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$\pm \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \pm \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\pm \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \pm \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right).$$
Question 2. Consider the map \( \eta: \mathbb{S}^2 \to \mathbb{R}^4 \) such that

\[
\eta(u, v, w) = (u^2 - v^2, uv, uw, vw),
\]

where all points \((u, v, w)\) on the sphere satisfy \(u^2 + v^2 + w^2 = 1\). Show that \( \eta(u, v, w) = \eta(u', v', w') \) if and only if \((u, v, w) = \pm(u', v', w')\), hence \( \eta \) defines a one-to-one map from \( \mathbb{R}P_2 \) to its image in \( \mathbb{R}^4 \). Show also that the image of \( \eta \) is a proper subset of \( F^{-1}(0) \) for the map \( F: \mathbb{R}^4 \to \mathbb{R}^2 \) such that

\[
F(x, y, z, t) = (y(z^2 - t^2) - xzt, y^2z^2 + y^2t^2 + z^2t^2 - yzt).
\]

**Proof.** Let \((u, v, w), (u', v', w') \in \mathbb{S}^2\). If \((u, v, w) = (u', v', w')\), then trivially \( \eta(u, v, w) = \eta(u', v', w') \). Also if \((u, v, w) = -(u', v', w')\), then

\[
\eta(u', v', w') = \eta(-u, -v, -w) = (u^2 - v^2, uv, uw, vw) = \eta(u, v, w).
\]

Conversely, suppose \( \eta(u, v, w) = \eta(u', v', w') \). Then we have the following equalities:

\[
\begin{align*}
uv &= u'v', \\
uw &= u'w', \\
vw &= v'w'.
\end{align*}
\]

Notice that

\[
(u^2 - v^2)^2 + 4(vw)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = u^4 + 2u^2v^2 + v^4 = (u^2 + v^2)^2,
\]

whence from \(u^2 - v^2 = u'^2 - v'^2\) and \(uv = u'v'\) we obtain

\[
u^2 + v^2 = u'^2 + v'^2.
\]

Adding this to \(u^2 - v^2 = u'^2 - v'^2\) we get

\[
2u^2 = 2u'^2, \quad \Rightarrow \quad u = \pm u'.
\]

Therefore

\[
v^2 = v'^2, \quad \Rightarrow \quad v = \pm v'.
\]

Since \((u, v, w), (u', v', w') \in \mathbb{S}^2\), we have that

\[
u^2 + v^2 + w^2 = 1 = u'^2 + v'^2 + w'^2 = 1,
\]

and so

\[
w = \pm w'.
\]

Thus

\[
u = \alpha u', \quad v = \beta v', \quad w = \gamma w'
\]

for some \(\alpha, \beta, \gamma = \pm 1\).

We claim that one can always assume that \(\alpha = \beta = \gamma\). Consider two cases.

1) Suppose there are two non-zero coordinates, say \(u, v \neq 0\). Then the corresponding coefficients coincide \(\alpha = \beta\). Indeed,

\[
vw = u'v' = \alpha \beta v, \quad \Rightarrow \quad 1 = \alpha \beta, \quad \Rightarrow \quad \alpha = \beta.
\]

Now if \(w = 0\), then \(w' = w\) = 0, and so

\[
(u', v', w') = (\alpha u, \alpha v, 0) = \alpha \cdot (u, v, 0) = \alpha \cdot (u, v, w).
\]

If \(w \neq 0\), then \(\alpha = \beta = \gamma\).

2) Suppose two of coordinates \((u, v, w)\) are zero, say, let \(v = w = 0\), and \(u \neq 0\). Then \(v' = w' = 0\), and \(u' = \pm u\), so
Thus \((u', v', w') = (\pm u, 0,0) = \pm (u, v, w)\).

Thus \(\eta(u, v, w) = \eta(u', v', w')\) if and only if \((u, v, w) = \pm (u', v', w')\).

Let us prove that the image of \(\eta\) is a proper subset of \(F^{-1}(0)\) for the map \(F: \mathbb{R}^4 \to \mathbb{R}^2\) defined by
\[
F(x, y, z, t) = (y(z^2 - t^2) - xzt, y^2z^2 + y^2t^2 + z^2t^2 - yzt).
\]

It suffices to prove that \(F \circ \eta: S^2 \to \mathbb{R}^2\) a constant map equal to \(0\). Indeed, since \(u^2 + v^2 + w^2 = 1\), we obtain that
\[
F \circ \eta(u, v, w) = F(u^2 - v^2, uv, uw, vw)
= (uv((uw)^2 - (vw)^2) - (u^2 - v^2)uvw,
   (uv)^2(uw)^2 + (uv)^2(vw)^2 + (uw)^2(vw)^2 - uwuvwv)
= (u^3vw^2 - uv^3w^2 - u^3vw^2 + uv^3w^2,
   u^4v^2w^2 + u^2v^4w^2 + u^2v^2w^4 - u^2v^2w^4)
= (0, (v^2 + u^2 + w^2 - 1)u^2v^2w^2) = (0, 0).
\]