Problem 1. Establish the relation between the E-wave and H-wave amplitudes.

Solution

For vacuum \((j = 0, \rho = 0, \mu = 1, \epsilon = 1)\) Maxwell equations can be written as:

\[
curl \vec{H} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\]

\[
curl \vec{E} = - \frac{1}{c} \frac{\partial \vec{H}}{\partial t}
\]

\[
div \vec{E} = 0
\]

\[
div \vec{H} = 0
\]

These equations can be reduced, e.g. for \(\vec{E}\) (and equivalent for \(\vec{H}\))

\[
curl \curl \vec{E} = \nabla \cdot \nabla \vec{E} - \nabla^2 \vec{E} = - \frac{1}{c} \curl \frac{\partial \vec{H}}{\partial t} = - \frac{1}{c} \frac{\partial \curl \vec{H}}{\partial t}
\]

\[
= - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}
\]

So:

\[
\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \nabla^2 \vec{E}
\]

Similarly:

\[
\frac{\partial^2 \vec{H}}{\partial t^2} = c^2 \nabla^2 \vec{H}
\]

Solutions for this wave equations that describe EM waves are:

\[
\vec{E} (\vec{r}, t) = \vec{a}_E \cos(\vec{k}_E \cdot \vec{r} - \omega t + \delta_E)
\]

\[
\vec{H} (\vec{r}, t) = \vec{a}_H \cos(\vec{k}_H \cdot \vec{r} - \omega t + \delta_H)
\]

From Maxwell’s equations follow also the relations

\[
\vec{E} = -\vec{s} \times \vec{H}
\]

\[
\vec{H} = \vec{s} \times \vec{E}
\]
expressing that the three vectors $\vec{E}$, $\vec{H}$, and $\vec{s}$ form a right-handed orthogonal triad of vectors. Thus we can choose the $z$-axis in the propagation direction $\vec{s}$, so that there are only electric and magnetic field components in the $x$- and $y$-direction. The end point of the electric and magnetic vectors is then described by:

$$E_x(z,t) = a_x \cos(k_n \cdot z - \omega t + \delta_x)$$
$$E_y(z,t) = a_y \cos(k_n \cdot x - \omega t + \delta_y)$$
$$H_x(z,t) = E_y(z,t)$$
$$H_y(z,t) = E_x(z,t)$$

So for vacuum:

$$\frac{E_x(z,t)}{H_y(z,t)} = 1$$

**Problem 2.** Show that in a weakly conducting medium, an electromagnetic wave gets rapidly attenuated with distance.

**Solution**

Considering the propagation of an electromagnetic wave through a conducting medium which obeys Ohm's law:

$$\vec{j} = \sigma \vec{E}$$

Here, $\sigma$ is the conductivity of the medium in question. Maxwell's equations for the wave take the form:

$$\text{curl } \vec{H} = \mu_0 j + \epsilon \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$
$$\text{curl } \vec{E} = - \frac{\partial \vec{H}}{\partial t}$$
$$\text{div } \vec{E} = 0$$
$$\text{div } \vec{H} = 0$$
where $\varepsilon$ is the dielectric constant of the medium. It follows, from the above equations, that

$$\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\sigma\vec{E} + \varepsilon\varepsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Looking for a wave-like solution of the form

$$\vec{E} = E_0 e^{i(kz - \omega t)}$$

we obtain the dispersion relation

$$k^2 = \mu_0 \omega (\varepsilon \varepsilon_0 \omega + i \sigma)$$

Consider a "weak" conductor for which $\varepsilon \varepsilon_0 \omega >> \sigma$. In this case, the dispersion relation yields

$$k \approx n \frac{\omega}{c} + i \frac{\sigma}{2} \sqrt{\frac{\mu_0}{\varepsilon \varepsilon_0}}$$

Substitution into wave equation gives:

$$\vec{E} = E_0 e^{-\frac{z}{d}} e^{i\left(\omega\left(\frac{\sqrt{\varepsilon}}{c} z - t\right)\right)}$$

Where

$$d = \frac{2}{\sigma} \sqrt{\frac{\varepsilon \varepsilon_0}{\mu_0}}$$

**Problem 4.** What is skin effect? What is skin depth? What information and what parameters would you need to determine skin depth of a given medium?

**Solution**

Skin effect is the phenomenon when an alternating current tends to concentrate in the outer layer of a conductor, caused by the self-induction of the conductor and resulting in increased resistance.
As it was shown in problem 2, the dispersion relation for EM wave in conducting material is:

\[ k^2 = \mu_0 \omega (\varepsilon \varepsilon_0 \omega + i \sigma) \]

In problem 2 we conclude that the amplitude of an electromagnetic wave propagating through a conductor decays exponentially on some length-scale, \( d \), which is termed the skin-depth. As it is seen from problem 2 solution the skin-depth for a poor conductor is independent of the frequency of the wave.

Considering a "good" conductor for which \( \sigma \gg \varepsilon \varepsilon_0 \omega \). In this case, the dispersion relation yields

\[ k \approx \sqrt{i \mu_0 \sigma \omega} \]

Substitution into solution of wave equation gives:

\[ \vec{E} = E_0 e^{-\frac{z}{d}} e^{i \omega (\sqrt{\varepsilon} z - t)} \]

Where

\[ d = \sqrt{\frac{2}{\mu_0 \sigma \omega}} \]

It can be seen that the skin-depth for a good conductor decreases with increasing wave frequency.

**Problem 5.** Concerning electromagnetic waves in a weakly conduction dielectric, prove that the B-waves lags in phase being the E-wave and find an expression for this phase difference.

**Solution**

As it was shown in problem 2

\[ \vec{E} = E_0 e^{i(kz - \omega t)} \]

and the dispersion relation

\[ k^2 = \mu_0 \omega (\varepsilon \varepsilon_0 \omega + i \sigma) \]
Similarly for magnetic field:

\[ \vec{B} = B_0 e^{i(kz - \omega t)} \]

Using Maxwell’s equations:

\[ \text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Which gives:

\[ \vec{k} \times \vec{E}_0 = \omega \vec{B} \]

Or

\[ \vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \]

So in a conductor, the complex phase of \( \vec{k} \) gives a phase difference between the electric and magnetic fields. This phase difference is given by the phase angle \( \phi \) of \( \vec{k} \):

\[ \tan \phi = -\frac{\text{Im}(\vec{k})}{\text{Re}(\vec{k})} \]

**Problem 6.** Calculate the time averaged energy density of an electromagnetic wave in a weakly conducting medium.

**Solution**

The power per unit volume dissipated via ohmic heating in a conducting medium takes the form

\[ P = \vec{j} \cdot \vec{E} = \sigma E^2. \]

Consider an electromagnetic wave of the form

\[ \vec{E} = E_0 e^{-\frac{z}{d}} e^{i\omega(\sqrt{\varepsilon} z - t)} \]
The mean power dissipated per unit area in the region \( z > 0 \) is written \[ \langle P \rangle = \frac{1}{2} \int_0^\infty \sigma E_0^2 e^{-\frac{z}{\mu_0}} \, dz = \frac{d\sigma}{4} E_0^2 = \sqrt{\frac{\sigma}{8 \mu_0 \omega}} E_0^2, \]

for a good conductor.

Now, according to equation

\[ u = \frac{|E|^2}{\mu_0 \omega} \text{Re}(k). \]

the mean electromagnetic power flux into the region \( z > 0 \) takes the form

\[ \langle u \rangle = \left( \frac{E \times B \cdot \hat{z}}{\mu_0} \right)_{z=0} = \frac{1}{2} \left( \frac{E_0^2}{\mu_0} k_t \right) \sqrt{\frac{\sigma}{8 \mu_0 \omega}} E_0^2. \]

It is clear from a comparison of the previous two equations, that all of the wave energy which flows into the region \( z > 0 \) is dissipated via ohmic heating.

**Problem 7.** Concerning EM waves in a weakly conducting medium, find an expression for the phase velocity of the wave.

**Solution**

From solutions of problems above for weakly conducting medium:

\[ \vec{E} = E_0 e^{-\frac{z}{\mu_0}} e^{i\omega \left( \frac{\sqrt{\varepsilon}}{c} z - t \right)} \]

The phase velocity is the velocity of a point that stays in phase with the wave,

For point staying at a fixed phase, we must have:

\[ \omega \left( \frac{\sqrt{\varepsilon}}{c} z - t \right) = \text{const} \]

\[ \omega \frac{\sqrt{\varepsilon}}{c} z = \omega t + \text{const} \]

So the phase velocity is given by:

\[ v_p = \frac{dz}{dt} = \frac{c}{\sqrt{\varepsilon}} \]