Task 1 – Analysing the sensor circuit
(LO 3: 3.1 part)
The two sensors are the same, but it is suspected that they have slightly different characteristics. Two tests were carried out to find out how much voltage they give out and how much internal resistance they have. Figure 1 shows the equivalent circuit of each sensor, showing the internal voltage source (VS), the internal resistance (R1), a connecting resistor (R2) and a load resistor for the tests (R3).

Solution.
Consider the first test, for which the switch is open and 0 mA. We see that the voltmeter is ideal (i.e. its internal resistance is equal to infinity) because 0 mA. According the Kirchhoff’s Voltage Law, write down the following equation:
\[ IR_1 + IR_2 + V = VS \iff VS = V. \]
Thus, the internal voltage sources of the sensors are \( VS_1 = 6.5 \text{ V} \) and \( VS_2 = 6.2 \text{ V} \).

Consider the second test, for which the switch is closed. To find the resistance \( R_1 \), we suppose that the ammeter is ideal (i.e. its internal resistance is equal to zero). We have in accordance with Kirchhoff’s Voltage Law:
\[ IR_1 + IR_2 + IR_3 = VS \iff R_1 = \frac{VS - I(R_2 + R_3)}{I} = \frac{VS}{I} - R_2 - R_3. \]

Find the internal resistance \( R_{11} \) of the first sensor:

<table>
<thead>
<tr>
<th>sensor</th>
<th>switch open</th>
<th>switch closed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I (mA)</td>
<td>V (V)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>6.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>6.2</td>
</tr>
</tbody>
</table>

Table 2: Sensor test results

Use the results in Table 2, along with Ohm’s law and Kirchhoff’s laws, to determine the internal voltage source (VS), and the internal resistance (R1) to the nearest 1 \( \Omega \), for both sensors. (3.1 part)
Find the internal resistance $R_1$ of the second sensor:

$$R_1 = \frac{V_{S_2}}{I} - R_2 - R_3 = \frac{6.2}{9.01 \times 10^{-3}} - 120 \Omega - 550 \Omega = 18.124 \Omega \approx 18 \Omega$$

Task 2 – Analysing the effect of connecting the two sensors in parallel  
(LO 3: 3.1 part, 3.2)

It was now decided to see if the sensors could be paralleled together, in order to supply more current. Figure 2 shows the equivalent circuit of the two sensors connected together in parallel.

First use Kirchhoff’s laws and mesh analysis to determine the current supplied from each sensor, and the total current and voltage developed in the load resistor R5.

Next, create a Thevenin equivalent circuit of the circuit in Figure 2 (the output terminals are A-B and R5 is considered as the load resistor), and verify that this new circuit develops the same current and voltage in the 250 Ω load resistor as the original circuit in Figure 2.

Now use this Thevenin equivalent circuit to determine the current and voltage which would be developed in load resistors of (i) 10 Ω and (ii) 10 kΩ, which would be connected between terminals A and B in place of R5. What value load resistor would lead to maximum power being transferred from the sensors to the load?  
(3.1 part, 3.2)

Solution.

Write the system of equations according Kirchhoff’s Voltage Law (see Figure 2-1):

$$\begin{align*}
I_1R_1 + I_2R_2 + I_3R_5 &= V_1; \\
I_2R_4 + I_3R_3 + I_4R_5 &= V_2.
\end{align*}$$

Express the current $I_3$ through the mesh currents $I_1$ and $I_2$ using Kirchhoff’s Current Law:

$$I_3 - I_1 - I_2 = 0 \iff I_3 = I_1 + I_2.$$  

Plug $I_3$ into the system of equations:

$$\begin{align*}
I_1R_1 + I_2R_2 + (I_1 + I_2)R_5 &= V_1; \\
I_2R_4 + I_3R_3 + (I_1 + I_2)R_5 &= V_2; \\
I_4R_5 + I_3(R_3 + R_4 + R_5) &= V_2.
\end{align*}$$
Plug the numerical values into the system of equation. We have:

\[
\begin{align*}
I_1(15+120+250)+250I_2 &= 6.5; \\
250I_1 + I_2(120+18+250) &= 6.2; \\
\Rightarrow 385I_1 + 250I_2 &= 6.5; \\
250I_1 + 388I_2 &= 6.2.
\end{align*}
\]

Solve the system by Cramer’s Rule:

\[
\begin{vmatrix}
6.5 & 250 \\ 6.2 & 388 \\ 385 & 250 \\ 250 & 388
\end{vmatrix} = 972 = 0.0112 \text{ A} = 11.2 \text{ mA}; \\
\begin{vmatrix}
385 & 6.5 \\ 250 & 6.2 \\ 385 & 250 \\ 250 & 388
\end{vmatrix} = 762 = 0.0088 \text{ A} = 8.8 \text{ mA}.
\]

Find the current \( I_3 \) through the load resistor:

\[ I_3 = I_1 + I_2 = 11.2 \text{ mA} + 8.8 \text{ mA} = 20 \text{ mA}. \]

Draw the Thevenin equivalent circuit for the output terminals A-B as shown in Figure 2-2.

Find the equivalent voltage for the Thevenin circuit \( V_{eq} \). This voltage is obtained at terminals A-B of the circuit with terminals A-B open circuited, as shown in Figure 2-3. According Kirchhoff’s Voltage Law, write the following equation for the circuit shown in Figure 2-3:
Use Kirchhoff’s Voltage Law again to find the equivalent voltage $V_{eq}$:

$$V_{eq} = I_a R_3 + I_b R_1 + V_2 = \frac{(V_1 - V_2)(R_3 + R_4)}{R_1 + R_2 + R_3 + R_4} + V_2 = \frac{V_1 R_3 + V_2 R_1 + V_2 R_2 + V_2 R_4}{R_1 + R_2 + R_3 + R_4} = 6.35 \text{ V}.$$ 

Calculate the output current $I_{ab}$ when the output terminals A and B are short circuited (see Figure 2-4). Using Kirchhoff’s Voltage Law twice, we have:

$$I_a R_1 + I_a R_2 + I_b R_3 + I_b R_4 = V_1 - V_2 \iff I_a = \frac{V_1 - V_2}{R_1 + R_2 + R_3 + R_4}.$$ 

Express the current $I_L$ according Kirchhoff’s Current Law:

$$I_L = I_a + I_b = \frac{V_1}{R_1 + R_2} + \frac{V_2}{R_3 + R_4} = 93.1 \text{ mA}.$$ 

Thus, the equivalent resistance $R_{eq}$ for the Thevenin circuit (see Figure 2-2) is:

$$R_{eq} = \frac{V_{eq}}{I_L} = \frac{6.35 \text{ V}}{93.1 \times 10^{-3} \text{ A}} = 68.2 \Omega.$$ 

According to Ohm’s Law, the current $I_3$ through the resistor $R_5$ is given by (see Figure 2-2):
\[ I_3 = \frac{V_{eq}}{R_{eq} + R_3} = \frac{6.35 \text{ V}}{68.2 \Omega + 250 \Omega} = 20.0 \text{ mA}. \]

As we can see, this result is the same as we got earlier.

Find \( I_3 \) for the load resistor \( R_3 = 10 \Omega \):
\[ I_3 = \frac{V_{eq}}{R_{eq} + R_3} = \frac{6.35 \text{ V}}{68.2 \Omega + 10 \Omega} = 81.2 \text{ mA}. \]

Find \( I_3 \) for the load resistor \( R_3 = 10 \text{k}\Omega \):
\[ I_3 = \frac{V_{eq}}{R_{eq} + R_3} = \frac{6.35 \text{ V}}{68.2 \Omega + 10000 \Omega} = 0.63 \text{ mA}. \]

Find the power \( p_3 \) transferred to the load \( R_3 \):
\[ p_3 = I_3^2 R_3 = \frac{V_{eq}^2 R_3}{(R_{eq} + R_3)^2}. \]

We can consider \( p_3 \) as a function of resistance \( R_3 \). The function \( p_3(R_3) \) reaches its maximum when its derivative equals zero:
\[ \frac{dp_3}{dR_3} = \frac{d}{dR_3} \left[ \frac{V_{eq}^2 R_3}{(R_{eq} + R_3)^2} \right] = V_{eq}^2 \left( \frac{(R_{eq} + R_3)^2 - 2(R_{eq} + R_3)R_3}{(R_{eq} + R_3)^4} \right) = 0; \ R_{eq} + R_3 - 2R_3 = 0; \ R_3 = R_{eq} = 68.2 \Omega. \]

Thus, when \( R_3 = 68.2 \Omega \), the power transferred to the load \( R_3 \) is maximal.